Rational Expressions Guide

What is a Rational Expression: A fraction in which the numerator (top) and/or denominator (bottom) are polynomials and the denominator is not equal to zero.

Evaluating Rational Expressions

Simply substitute the given value for x and use the order of operations to evaluate the final answer, i.e. evaluate the numerator and denominator separately and reduce.

Evaluate $\frac{3x+6}{2x-4}$ for $x=-2$.

$$\frac{3x+6}{2x-4}=\frac{3(-2)+6}{2(-2)-4}=\frac{-6+6}{-4-4}=\frac{0}{-8}=0$$

Finding Values That Make Rational Expressions Undefined

Set the denominator equal to zero and solve. Remember, x can't equal these values.

Example: Find any values for the variable for which $\frac{x+5}{x+2}$ is undefined.

$$x+2=0$$
$$-2 \quad -2$$
$$x=-2$$

So, when $x=-2$, the expression is undefined. Therefore, we say $x\neq-2$.

Example: Find any values for the variable for which $\frac{9a^2}{a^2-5a+6}$ is undefined.

$$a^2-5a+6=0$$
$$(a-2)(a-3)=0$$
$$a-2=0 \quad \text{or} \quad a-3=0$$
$$+2 \quad +2 \qquad\quad +3 \quad +3$$
$$a=2 \quad \text{or} \quad a=3$$

So, when $a=2$ or $a=3$, the expression is undefined. Therefore, we say $a\neq 2$ and $a\neq 3$.

Simplifying Rational Expressions

Steps:
1) Factor the numerator and denominator completely if you can.
2) Cancel common factors up/down where possible. What you have leftover is your answer.

$$\frac{2x^2-8}{2x^2+10x+12}=\frac{2(x^2-4)}{2(x^2+5x+6)}=\frac{2(x+2)(x-2)}{2(x+2)(x+3)}=\frac{x-2}{x+3}$$

Multiplying Rational Expressions

Steps:
1) Factor the numerators and denominators completely if you can.
2) Cancel common factors up/down and/or diagonally where possible.
3) Multiply the numerators, and multiply the denominators for your final answer.

Note: If everything cancels in the numerator, put a 1 in the numerator of your answer.

$$\frac{x^2+3x}{x^2-3x-4}\bullet\frac{x^2-5x+4}{x^2+2x-3}=\frac{x(x+3)}{(x-4)(x+1)}\bullet\frac{(x-4)(x-1)}{(x+3)(x-1)}=\frac{x}{x+1}$$

Dividing Rational Expressions

Change the division to multiplication and change the second fraction to its reciprocal (flip it). Then, follow the steps for multiplying rational expressions.

$$\frac{x^2-4}{x^2-1}\div\frac{2x^2+4x}{x-1}=\frac{x^2-4}{x^2-1}\bullet\frac{x-1}{2x^2+4x}=\frac{(x+2)(x-2)}{(x+1)(x-1)}\bullet\frac{x-1}{2x(x+2)}=\frac{x-2}{2x(x+1)}$$

Adding/Subtracting Rational Expressions

1) If the denominators are the same, skip to step 3. Otherwise, factor the denominators completely where possible.

2) Determine the LCD* and write new fractions with this LCD in the denominators.

   *To determine the LCD, every factor in the denominators must be represented in the LCD. For example, if one fraction has (x-2) in the denominator and the other has (x-2)(x+5) in the denominator, then the LCD would be (x-2)(x+5) because this contains every factor from our denominators. You don't have to list (x-2) twice. However, if the first denominator is (x+3) and the other is $(x+3)^2$, the LCD would be $(x+3)^2$ because although (x+3) would have been enough for representing the factors from the first fraction, it wouldn't be enough to represent $(x+3)^2$. As a rule, use the highest exponent when two factors are the same.

   Now that you know the new denominators, look to see how your denominators changed so that you can change the numerators in the same way. For example, if a fraction from the original problem had (x-2) in the denominator and it now has (x-2)(x+5), then you multiplied your denominator by (x+5). Therefore, you must also multiply your numerator of the original fraction by (x+5) for the new fraction. Your fractions now have been adjusted to have common denominators.

3) Combine the fractions together by keeping the same denominator and adding/subtracting the numerators. (Use the distributive property where applicable. Then if adding, simply combine like terms. If subtracting, make sure to distribute the minus sign to every term in the second fraction before combining like terms.)

4) Finally, simplify the rational expression. In other words, factor where you can, and cancel any common factors that you may have.

$$\frac{x}{x-2}-\frac{4x+6}{x^2+3x-10}=\frac{x}{x-2}-\frac{4x+6}{(x-2)(x+5)} \quad \text{(Step 1)}$$

$$=\frac{x(x+5)}{(x-2)(x+5)}-\frac{4x+6}{(x-2)(x+5)} \quad \text{(Step 2)}$$

$$=\frac{x(x+5)-(4x+6)}{(x-2)(x+5)}=\frac{x^2+5x-4x-6}{(x-2)(x+5)}=\frac{x^2+x-6}{(x-2)(x+5)} \quad \text{(Step 3)}$$

$$=\frac{(x-2)(x+3)}{(x-2)(x+5)}=\frac{x+3}{x+5} \quad \text{(Step 4)}$$

Complex Fractions

A complex fraction is a quotient with one or more fractions in the numerator, or denominator, or both.

If the complex fraction contains only one term in the numerator and only one term in the denominator, then rewrite the problem using the ÷ symbol, i.e. fraction ÷ fraction, and then follow the rules for "Dividing Rational Expression" (see front).

$$\frac{\frac{1}{x+5}}{\frac{4}{x^2-25}}=\frac{1}{x+5}\div\frac{4}{x^2-25}=\frac{1}{x+5}\bullet\frac{x^2-25}{4}=\frac{1}{x+5}\bullet\frac{(x+5)(x-5)}{4}=\frac{(x-5)}{4}$$

Otherwise, find the LCD of all fractions within the complex fraction and multiply all terms in the numerator and denominator of the complex fraction by this LCD. This will reduce the problem to one fraction that you can simplify by using the distributive property and then combining like terms where possible. Finally, consider the steps of "Simplifying Rational Expressions" in case your fraction could be simplified further.

$$\frac{\frac{1}{x}+\frac{2}{x+2}}{\frac{4}{x}-\frac{3}{x+2}} \quad \boxed{\text{LCD: } x(x+2)} \rightarrow \frac{x(x+2)\left[\frac{1}{x}+\frac{2}{x+2}\right]}{x(x+2)\left[\frac{4}{x}-\frac{3}{x+2}\right]}=\frac{x(x+2)\frac{1}{x}+x(x+2)\frac{2}{x+2}}{x(x+2)\frac{4}{x}-x(x+2)\frac{3}{x+2}}=\frac{1(x+2)+2x}{4(x+2)-3x}=\frac{x+2+2x}{4x+8-3x}=\frac{3x+2}{x+8}$$

Solving Equations with Rational Expressions

1) Factor denominators completely where needed.

2) Determine the LCD (see * on back of page 1), and then multiply ALL TERMS in the equation by this LCD. Remember, a fraction is considered one term. This will cancel out the denominators leaving an equation with no fractions.

3) Then use the distributive property where needed and solve the equation using the normal rules**.
   **If your equation at this point is linear, i.e. only contains x^1, then combine like terms on sides separately and use opposite operations to get x on one side of the equation and a number on the other side. If your equation at this point is quadratic, i.e. contains x^2, then get all terms on one side of the equation so that 0 is on the other side. If the quadratic will factor, then factor and set each factor equal to 0 and solve. If the quadratic can't be factored, then use the quadratic formula or the completing the square method to find your solution(s).

4) Check each proposed solution by substituting it into the original equation. Any solution that causes a denominator in the original equation to equal 0 are called extraneous solutions and should be rejected.

NOTE: If your variables disappear during the solving process and only numbers remain, then determine if the statement you're left with is true or false. If the statement is true, like 2=2, then we say the solution is "all real numbers except x=#" and we write the set notation $\{x \mid x \neq \#\}$ where # is any real number that would make a denominator in the original equation equal 0. If the statement is false, like 2=5, then we say the equation has "no solution" and we write $\varnothing$.

Example 1:

$$\frac{4}{3x+6}-\frac{3}{x+3}=\frac{8}{x^2+5x+6} \rightarrow \frac{4}{3(x+2)}-\frac{3}{x+3}=\frac{8}{(x+2)(x+3)} \quad \text{LCD: } 3(x+2)(x+3)$$

$$\rightarrow \frac{3(x+2)(x+3)}{}\bullet\frac{4}{3(x+2)}-\frac{3(x+2)(x+3)}{}\bullet\frac{3}{x+3}=\frac{3(x+2)(x+3)}{}\bullet\frac{8}{(x+2)(x+3)}$$

$$\rightarrow 4(x+3)-9(x+2)=24$$

$$\rightarrow 4x+12-9x-18=24$$

$$\rightarrow -5x-6=24$$

$$+6 \quad +6$$

$$\rightarrow \frac{-5x}{-5}=\frac{30}{-5}$$

$x=-6$ If you substitute -6 back into the original equation, it works.

Example 2:

$$\frac{3x}{x^2+5x+6}=\frac{5x}{x^2+2x-3}-\frac{2}{x^2+x-2} \rightarrow \frac{3x}{(x+2)(x+3)}=\frac{5x}{(x+3)(x-1)}-\frac{2}{(x+2)(x-1)} \quad \text{LCD: } (x-1)(x+2)(x+3)$$

$$\rightarrow (x-1)(x+2)(x+3)\frac{3x}{(x+2)(x+3)}=(x-1)(x+2)(x+3)\frac{5x}{(x+3)(x-1)}-(x-1)(x+2)(x+3)\frac{2}{(x+2)(x-1)}$$

$$\rightarrow 3x(x-1)=5x(x+2)-2(x+3)$$

$$\rightarrow 3x^2-3x=5x^2+10x-2x-6$$

$$\rightarrow 3x^2-3x=5x^2+8x-6 \quad \text{QUADRATIC!}$$

$$-3x^2 \qquad -3x^2$$

$$\rightarrow -3x=2x^2+8x-6$$

$$+3x \qquad +3x$$

$$\rightarrow 0=2x^2+11x-6$$

$$\rightarrow 0=(2x-1)(x+6)$$

$$\rightarrow 2x-1=0 \qquad x+6=0$$

$$+1 \quad +1 \qquad -6 \quad -6$$

$$\frac{2x}{2}=\frac{1}{2} \qquad x=-6$$

$x=\frac{1}{2}$ If you substitute -6 into the original equation it works, and the same is true for $\frac{1}{2}$.

Helpful YouTube videos:

1) https://www.youtube.com/watch?v=7Uos1ED3KHI
2) https://www.youtube.com/watch?v=3GL69IA2q4s
3) https://www.youtube.com/watch?v=f-wz_ZzSDdg
4) https://www.youtube.com/watch?v=gcnk8TnzsLc
5) https://www.youtube.com/watch?v=y_DweTAEYWk
6) https://www.youtube.com/watch?v=3tmFTHOP6Pc
7) https://www.youtube.com/watch?v=6eqgIZyXgK8